Lecture 35

Exam Stuff

- Orthogonal, symmetric matrices } Know how to recognize those
- Coordinates

- If you add 2 symmetric matrices you get symmetric matrix
  ↳ does not work for multiplication.
- The opposite is true for orthogonal matrices.

- Coordinates

vectors $\vec{v}$ ↓ $[\vec{v}]_\beta$

linear transformation $T(\vec{x}) = A\vec{x}$ ↓

$A = [T]$

$$\boxed{\begin{aligned} A &= S_\beta [T]_\beta S_\beta^{-1} \\ [T_\beta] &= S_\beta^{-1} A S_\beta \end{aligned}}$$ *

$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$, e-vecs: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

e-vals: 1, 2

$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ $\quad [\vec{v}]_\beta = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$ $\quad$ so $\vec{v} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 12\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A\vec{v} = 1 \cdot 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot 12 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$[A\vec{v}]_\beta = \begin{bmatrix} 7 \\ 24 \end{bmatrix}$

- Whats lost by considering $[\vec{v}]_\beta$ instead of $\vec{v}$? $[\vec{v}]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$[\vec{w}]_\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so $\vec{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Notes for today

$\boxed{\text{E.g.}}$ Suppose all ND students return to campus happy at the end of August. Suppose further that there's 25% chance every day that a student who goes to bed happy wakes up sad, but also a 50% chance that a sad student wakes up happy. What fraction of ND students go home sad for Christmas?

Soln: Let $\vec{v}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$ be the vector whose coordinates represent the fractions of students $x_1(n)$, $x_2(n)$ who are happy or sad.

Then $x_1(n) + x_2(n) = 1 \quad \forall n$.

Also $\vec{v}(n+1) = \underbrace{\begin{bmatrix} .75 & .50 \\ .25 & .50 \end{bmatrix}}_{A} \vec{v}(n)$

- We must compute $\vec{v}(100)$ from $\vec{v}(0)$

So $\vec{v}(100) = A^{100}\vec{v}(0) = A^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Def: A vector $\vec{v}$ (or matrix A) is called non-negative if all its entries are ≥ 0.

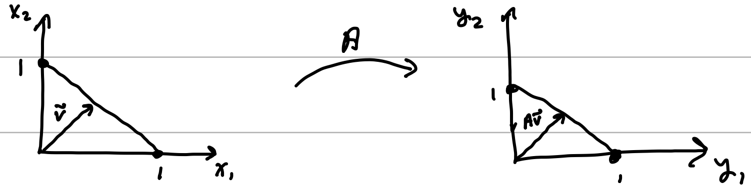
$\vec{v}/A$ is positive if all of its entries are positive.

Def: A <u>distribution vector</u> $\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is a non-negative vector such that $x_1 + \ldots + x_n = 1$

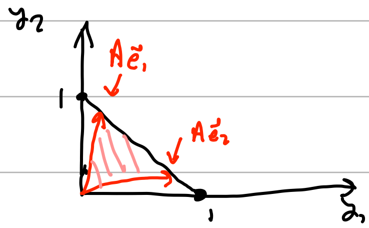
A <u>transition</u> matrix $A \in M_{n \times n}$ is a matrix whose columns are distribution vectors.

FACTs - If $A \in M_{n \times n}$ is a transition matrix and $\vec{v} \in \mathbb{R}^n$ is a distribution vector, then

- All powers A^k are transition matrices
- $A\vec{v}$ is a distribution vector.



but here the standard basis vecs don't map to standard basis vecs. It looks more like this:



- After 100 days we get $[0.67, 0.33]$, so 67% people are happy and 33% people are sad.

- He pointed out after some x days, the distribution vector does not change
  - ↳ Even if you change the initial distribution vector, the distribution vector after x days is the EXACT SAME.

E-vals	1	1/4
E-vecs	$\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

- Take our initial vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
  - Need $c_1 > 0$

$A^{100}\vec{v} = c_1 A^{100}\begin{pmatrix}2/3\\1/3\end{pmatrix} + c_2 A^{100}\begin{pmatrix}-1\\1\end{pmatrix}$

$= c_1\begin{pmatrix}2/3\\1/3\end{pmatrix} + c_2\left(\frac{1}{4}\right)^{100}\begin{pmatrix}-1\\1\end{pmatrix} \approx c_1\begin{pmatrix}2/3\\1/3\end{pmatrix}$ for n that is large

<u>Peron - Ferbenius Thm</u> : Given non-negative $\vec{v} \in \mathbb{R}^n$ and $A \in M_{n\times n}$, suppose some power of A^k of A is positive. Then $\exists$ positive eigenvector $\vec{v}$ for A with e-val $\lambda > 0$ and $\forall$ non-negative vector $\vec{w}$ we have $\lim_{n\to\infty} \frac{A^n\vec{w}}{\|A^n\vec{w}\|} = \vec{v}$

<u>Rmk</u>: If A is a transition matrix then $\lambda = 1$ and can take $\vec{v}$ to be distribution vector.

<u>Morally Speaking</u>

λ is the unique largest eigenvalue of A.